ON THE UNIFICATION OF QUANTUM 3–MANIFOLD INVARIANTS

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Abstract. In 2006 Habiro initiated a construction of generating functions for Witten–Reshetikhin–Turaev (WRT) invariants known as unified WRT invariants. In a series of papers together with Irmgard Bühler and Christian Blanchet we extended his construction to a larger class of 3–manifolds. The unified invariants provide a strong tool to study properties of the whole collection of WRT invariants, e.g. their integrality, and hence, their categorification.

In this paper we give a survey on ideas and techniques used in the construction of the unified invariants.

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Introduction

Background. In the 60s and 70s Rochlin, Lickorish and Kirby established a remarkable connection between links and 3–manifolds. They showed that every 3–manifold can be obtained by surgery (on S^3) along framed links, and surgeries along two links give the same 3–manifold if and only if the links are related by a sequence of Kirby moves. This allows us to think of 3–manifolds as equivalence classes of framed links modulo the relation generated by Kirby moves.

After the discovery of the Jones polynomial in 1984, knot theory experienced the transformation from an esoteric branch of pure mathematics to a dynamic research field with deep connections to mathematical physics, the theory of integrable and dynamic systems, von Neumann algebras, representation theory, homological algebra, algebraic geometry, etc. Among important developments were the constructions of *quantum link invariants* (generalization of the Jones polynomial), and of *the Kontsevich integral* (universal finite type invariant).

The quantum link invariants were extended to 3–manifolds by Witten and Reshetikhin–Turaev (WRT), the lift of the Kontsevich integral to 3–manifolds was defined by Le–Murakami–Ohtsuki (LMO). However, while the relationship between the Kontsevich integral and quantum link invariants is simple, the relationship between the LMO invariant and the WRT invariants is much more complicated and remains mysterious in many cases.

Let us explain this in more details. The Kontsevich integral of a knot takes values in a certain algebra $\mathcal{A}(S^1)$ of chord diagrams. Any semi–simple Lie algebra and its module define a map $\mathcal{A}(S^1) \to \mathbb{Q}[[h]]$ called a weight system. One important result of Le–Murakami and Kassel is that the following diagram commutes, where the $\mathfrak{sl}_2$ weight system uses the 2–dimensional defining representation.

Key words and phrases. 3–manifold, unified quantum invariant, Habiro ring, Andrews identity, Frobenius map.

$$\begin{tikzcd}
& \mathbb{Z}[q^{\pm 1}] \arrow[dr, "q=e^h"] & \\
\{\text{knots}\} \arrow[ur, "\text{Jones polynomial}"] \arrow[dr, "\text{Kontsevich integral}"'] & & \mathbb{Q}[[h]] \\
& \mathcal{A}(S^1) \arrow[ur, "\mathfrak{sl}_2\text{-weight system}"'] &
\end{tikzcd}$$

In particular, this proves that the Kontsevich integral dominates the Jones polynomial, and similarly all quantum link invariants coming from Lie algebras. Hence in addition to being universal for finite type, the Kontsevich integral is also universal for all quantum link invariants. It is conjectured that the Kontsevich integral separates knots.

Does there exist a similar commutative diagram for 3–manifolds? The quantum WRT invariant associates with a compact orientable 3–manifold M, a root of unity ξ and a semi–simple Lie algebra, say $\mathfrak{sl}_2$ for simplicity, a complex number $\tau_M(\xi)$. The LMO invariant takes values in a certain algebra $\mathcal{A}$ of Feynman diagrams. Every semi–simple Lie algebra defines a weight system map from $\mathcal{A}$ to $\mathbb{Q}[[h]]$. Hence, we have

$$\begin{tikzcd}
& \mathbb{C} & \\
\{\text{3–manifolds}\} \arrow[ur, "\tau_M(\xi)"] \arrow[dr, "\text{LMO}"'] & & \mathbb{Q}[[h]] \arrow[ul, dotted] \\
& \mathcal{A} \arrow[ur, "\mathfrak{sl}_2\text{–weight system}"'] &
\end{tikzcd}$$

The image of the composition of the two bottom arrows is known as the Ohtsuki series [20]. Ohtsuki showed that for any rational homology 3–sphere M and a root of unity ξ of *prime* order p, the first $(p-1)/2$ coefficients, modulo p, of the Ohtsuki series are determined by $\tau_M(\xi)$. This is shown by the dotted arrow in the diagram.

This result of Ohtsuki raises many interesting questions: Does there exist any relationship between the LMO and WRT invariants at roots of unity of non–prime order? Is the whole set of WRT invariants determined by the LMO invariant? The discovery of the series prompted Ohtsuki to build the theory of finite type invariants of homology 3–spheres, which was further developed by Goussarov and Habiro.

In the 90s, Habegger, Garoufalidis and Beliakova showed that the LMO invariant is trivial if the first Betti number of a 3–manifold is bigger than 3, and for the Betti numbers 1,2 and 3, the LMO invariant is determined by the classical Casson–Walker–Lescop invariant. However, in the case of rational homology 3–spheres, the LMO invariant is more powerful, it is a universal finite type invariant in the Goussarov-Habiro sense.

The relationship between the LMO and WRT invariants at non–prime roots of unity remained open for quite a while. This is because all known techniques heavily rely on the fact that the order of the root is prime and can not be extended to other roots.

However, recently Habiro's theory [7] of unified invariants provided a complete solution of this problem in the case of integral homology 3–spheres. In this case, also the question of integrality of the WRT invariants at non–prime roots of unity was solved simultaneously. Though intensively studied (see [17], [16], [6], [14] and the references there), the integrality of WRT invariant was previously known for prime roots of unity only. Note that a conceptual solution of the integrality problem is of primary importance for any attempt of categorification of the WRT invariants (compare [9]).

Unified invariants of integral homology 3–spheres. The unification of the WRT invariants was initiated in 2006 by Habiro. For any integral homology 3–sphere M, Habiro [7] constructed a *unified invariant* J_M whose evaluation at any root of unity coincides with the value of the WRT invariant at that root. Habiro's unified invariant J_M is an element of the following ring (Habiro's ring)

$$\widehat{\mathbb{Z}[q]} := \lim_{\overleftarrow{k}} \frac{\mathbb{Z}[q]}{((q;q)_k)}, \qquad \text{where} \quad (q;q)_k = \prod_{j=1}^{k}(1-q^j).$$

Every element $f(q) \in \widehat{\mathbb{Z}[q]}$ can be written (non–uniquely) as an infinite sum

$$f(q) = \sum_{k\geq 0} f_k(q)\,(1-q)(1-q^2)...(1-q^k),$$

with $f_k(q) \in \mathbb{Z}[q]$. When $q = \xi$, a root of unity, only a finite number of terms on the right hand side are not zero, hence the evaluation $\mathrm{ev}_\xi(f(q))$ is well–defined and is an algebraic integer. However, the fact that the unified invariant belongs to the Habiro ring is stronger than just integrality of $\tau_M(\xi)$.

The Habiro ring has beautiful arithmetic properties. Every element $f(q) \in \widehat{\mathbb{Z}[q]}$ can be considered as a function whose domain is the set of roots of unity. Moreover, there is a natural Taylor series for f at every root of unity. In [7] it is shown that two elements $f, g \in \widehat{\mathbb{Z}[q]}$ are the same if and only if their Taylor series at a root of unity coincide. In addition, each function $f(q) \in \widehat{\mathbb{Z}[q]}$ is totally determined by its values at, say, infinitely many roots of order 3^n, $n \in \mathbb{N}$. Due to these properties the Habiro ring is also called a ring of "analytic functions at roots of unity". Thus belonging to $\widehat{\mathbb{Z}[q]}$ means that the collection of the WRT invariants is far from a random collection of algebraic integers; together they form a nice function.

General properties of the Habiro ring imply that for any integral homology 3–sphere M, the Taylor expansion of the unified invariant J_M at $q = 1$ coincides with the Ohtsuki series and dominates WRT invariants of M at all roots of unity (not only of prime order). This is summarized in the following commutative diagram.

$$\begin{tikzcd}
 & \mathbb{C} & \\
\{\mathbb{Z}\mathrm{HS}\} \arrow[ur, "\tau_M(\xi)"] \arrow[r, "J_M(q)"] \arrow[dr, "\mathrm{LMO}"'] & \widehat{\mathbb{Z}[q]} \arrow[u, "q=\xi"'] \arrow[r, hook] & \mathbb{Z}[[1-q]] \arrow[d, hook, "h=1-q"] \\
 & \mathcal{A} \arrow[r, "\mathfrak{sl}_2\text{–weight}", "\text{system}"'] & \mathbb{Q}[[h]]
\end{tikzcd}$$

By $\mathbb{Z}$HS we denoted here the set of integral homology 3–spheres. In particular, this shows that Ohtsuki series has integral coefficients (which was conjectured by Lawrence and first proved by Rozansky).

Recently, Habiro ring found applications in algebraic geometry for constructing varieties over the non–existing field of one element [15].

Unified invariants of rational homology 3–spheres. In [2], we give a full generalization of the Habiro theory to rational homology 3–spheres. This requires completely new techniques coming from number theory, commutative algebra, quantum group and knot theory. Let us explain this in more details.

Assume M is a rational homology 3–sphere with $|H_1(M, \mathbb{Z})| = b$, where for a finite group G we denote by $|G|$ the number of its elements. Then our unified invariant I_M belongs to a modification $\mathcal{R}_b$ of the Habiro ring where b is inverted. Unlike the case $b = 1$, the modified Habiro ring is not an integral domain, but a direct product of integral domains, where each factor is determined by its own Taylor expansion at some root of unity. There is a decomposition $I_M = \prod_{c|b} I_{M,c}$, where $I_{M,c}$ dominates the set $\{\tau_M(\xi)|(\mathrm{ord}(\xi), b) = c\}$. If $b = 1$, then I_M coincides with Habiro's J_M. The invariant $I_{M,1}$ was first defined in [12].

Our results can be summarized in the following commutative diagram. Here we assume for simplicity that $b = p^k$ is a power of a prime and put $e_n := \exp(2\pi I/n)$ the primitive nth root of unity.

$$\begin{tikzcd}
 & \mathbb{C} & \\
 & \mathcal{R}_b \arrow[u, "q=\xi"'] \arrow[r, hook] & \displaystyle\prod_{i=0}^{\infty} \mathbb{Z}\left[\frac{1}{p}, e_{p^i}\right][[e_{p^i} - q]] \arrow[dd, "\text{projection to } i=0"] \\
\{\mathbb{Q}\mathrm{HS}\} \arrow[uur, "\tau_M(\xi)"] \arrow[ur, "I_M(q)"'] \arrow[dr, "\mathrm{LMO}"'] & & \\
 & \mathcal{A} \arrow[r, "\mathfrak{sl}_2\text{–weight}", "\text{system}"'] & \mathbb{Q}[[h]]
\end{tikzcd}$$

By $\mathbb{Q}$HS we denote the set of rational homology 3–spheres M, with $|H_1(M, \mathbb{Z})| = b$. In particular, for any $M \in \mathbb{Q}$HS, we generalize Ohtsuki series as follows. Let us fix a divisor $c < b$ of b, then the Taylor expansion of I_M at e_c is a power series in $(q - e_c)$ with coefficients

in $\mathbb{Z}[1/b][e_c]$ which dominates the WRT invariants at roots of unity whose order has the greatest common divisor c with b. If $c = b = p^k$, then $I_{M,b}$ is a priory determined by a product of power series $\prod_{i\geq k} \mathbb{Z}\left[\frac{1}{p}, e_{p^i}\right]\left[\left[e_{p^i} - q\right]\right]$, however, conjecturally, it is enough to consider the series in $q - e_b$.

The commutative diagram tells us that LMO invariant determines $I_{M,1}$, or the set of WRT invariants at roots of unity coprime with b. Since there is no direct way to obtain power series in $(q - e_c)$ from the LMO invariant, we conjecture the existence of the refined universal finite type invariant, dominating our power series. On the physical level of rigor, this means that the new refined invariant should capture more than just the contribution of flat connections from the Chern–Simons theory.

The methods used in the unification of the WRT invariants also led to the full solution of the integrality problem for quantum $SO(3)$ and $SU(2)$ WRT invariants. In [4, 3], we showed that $\tau^G_M(\xi)$ for any 3–manifold M and any root of unity is always an algebraic integer. Here $G = SO(3)$ or $SU(2)$. The integrality of the spin and cohomological refinements is work in progress.

Assume M is the Poincaré homology 3-sphere, obtained by surgery on a left-hand trefoil with framing -1. Then

$$I_M = \frac{1}{1-q}\sum_{k=0}^{\infty} q^k (1-q^{k+1})(1-q^{k+2})\dots(1-q^{2k+1})\,.$$

We expect that the categorification of the WRT invariants will lead to a homology theory with Euler characteristic given by I_M.

The paper is organized as follows. In Section 1 we recall the definitions of Kirby moves, WRT invariants and of the cyclotomic expansion for the colored Jones polynomial. In Section 2 we state our main results and outline the proofs. Section 3 is devoted to the discussion of the rings, where the unified invariants take their values. In addition, we construct generalized Ohtsuki series. The Laplace transform method, the Andrews identity, Frobenius maps and the integrality of WRT invariants are explained in the last Section.

1. Quantum (WRT) invariants

1.1. **Notations and conventions.** We will consider $q^{1/4}$ as a free parameter. Let

$$\{n\} = q^{n/2} - q^{-n/2}, \quad \{n\}! = \prod_{i=1}^{n}\{i\}, \quad [n] = \frac{\{n\}}{\{1\}}.$$

We denote the set $\{1, 2, 3, \dots\}$ by $\mathbb{N}$. We also use the following notation from q–calculus:

$$(x; q)_n := \prod_{j=1}^{n}(1 - xq^{j-1}).$$

Throughout this paper, ξ will be a primitive root of unity of *odd* order r and $e_n := \exp(2\pi I/n)$.

All 3–manifolds in this paper are supposed to be closed and oriented. Every link in S^3 is framed, oriented, and has components ordered. For a link L, let (L_{ij}) be its linking

matrix, where for $i \neq j$, $L_{ij} := \mathrm{lk}(L_i, L_j)$ is the linking number of the ith and the jth components, and $L_{ii} = b_i$ is the framing of L_i, given by the linking number of L_i with its push off along the first vector in the framing.

Surgery along the framed link L consists of removing a tubular neighborhood of L from S^3 and then gluing it back with the diffeomorphism given by framing (i.e. the meridian of each removed solid torus, diffeomorhic to a neighborhood of L_i, is identified with the longitude of the complement twisted b_i times along the meridian). We denote by M the result of the surgery.

1.2. **The colored Jones polynomial.** Suppose L is a framed oriented link with m ordered components and $V_1, \dots, V_m$ are finite-dimensional modules over a ribbon Hopf algebra. Then one can define the quantum invariant $J_L(V_1, \dots, V_m)$ through the machinery of quantum link invariant theory, see [21]. The quantized enveloping algebra $U_h(sl_2)$ of sl_2 is a ribbon algebra, and for each positive integer n there is a unique $U_h(sl_2)$-module of dimension n. The quantum link invariant $J_L(n_1, \dots, n_m)$, where n_j stands for the n_j-dimensional $U_h(sl_2)$-module, is usually called the colored Jones polynomial, with the n_j's being the colors. When all the colors are 2, $J_L(2, \dots, 2)$ is the usual Jones polynomial, which can be defined using skein relation.

One can construct the colored Jones polynomial without the quantum group theory by first defining the Jones polynomial through the skein relation, and then defining the colored Jones polynomial by using cablings, see e.g. [10, 11].

Let us recall here a few well–known formulas. For the unknot U with 0 framing one has

$$J_U(n) = [n] \tag{1}$$

Moreover, J_L is multiplicative with respect to the disjoint union. If L_1 is obtained from L by increasing the framing of the ith component by 1, then

$$J_{L_1}(n_1, \dots, n_m) = q^{(n_i^2-1)/4} J_L(n_1, \dots, n_m). \tag{2}$$

If all the colors n_i are odd, then $J_L(n_1, \dots, n_m) \in \mathbb{Z}[q^{\pm 1}]$.

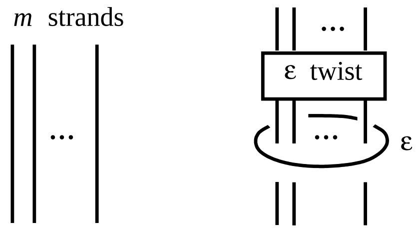
1.3. **Kirby moves.** By Kirby theorem, any link invariant which does not change under Kirby moves is an invariant of a 3–manifold given by surgery on that link. Let us first recall what the Kirby moves are.

K1–Move (handle slide): For some $i \neq j$, replace the ith component L_i with L_i', a band connected sum of L_i with a push off of L_j (defined by the framing), with $b_i' = b_i + b_j + 2\mathrm{lk}(L_i, L_j)$.

K2–Move (blow up): Add (or delete) a split unknotted component with framing ± 1.

These two moves are equivalent to the one Fenn–Rourke move defined as follows:

FR–Move Locally the following two pictures are interchangeable



where $\varepsilon \in \{1, -1\}$ and the closed component has framing ε. Note that K2–Move corresponds to the case when $m = 0$.

The main idea of the construction of the WRT invariants is to make the colored Jones polynomial invariant under Kirby moves by averaging over all colors. To make this precise, we have to choose the quantum parameter q to be a root of unity, otherwise the sum would be infinite.

1.4. **Evaluation and Gauss sums.** For each root of unity ξ of odd order r, we define the evaluation map ev_ξ by replacing q with ξ.

Suppose $f(q; n_1, \dots, n_m)$ is a function of variables $q^{\pm 1}$ and integers $n_1, \dots, n_m$. In quantum topology, the following sum plays an important role

$$\sum_{n_i}^{\xi} f := \sum_{\substack{0<n_i<2r \\ n_i \text{ odd}}} \mathrm{ev}_\xi f(q; n_1, \dots, n_m)$$

where in the sum all the n_i run over the set of *odd* numbers between 0 and $2r$.

In particular, the following variation of the Gauss sum

$$\gamma_b(\xi) := \sum_{n}^{\xi} q^{b\frac{n^2-1}{4}}$$

is well–defined, since for odd n, $4 \mid n^2 - 1$. It is known that, for odd r, $|\gamma_b(\xi)|$ is never 0.

1.5. **Definition of the WRT invariants.** Let

$$F_L(\xi) := \sum_{n_i}^{\xi} \prod_{i=1}^{m} [n_i] \; J_L(n_1, \dots, n_m).$$

Theorem 1. [Reshetikhin–Turaev] $F_L(\xi)$ *is invariant under K1–Move.*

An important special case is when $L = U^b$, the unknot with framing $b \neq 0$. In that case $F_{U^b}(\xi)$ can be calculated using the Gauss sum and is nonzero.

Let σ_+ (respectively σ_-) be the number of positive (negative) eigenvalues of the linking matrix of L. Then we define

$$\tau_M(\xi) = \frac{F_L(\xi)}{(F_{U^{+1}}(\xi))^{\sigma_+} \, (F_{U^{-1}}(\xi))^{\sigma_-}} . \tag{3}$$

It is easy to see that $\tau_M(\xi)$ is invariant under K2–Move, and hence, by Theorem 1, it is a topological invariant of M called the $SO(3)$ WRT invariant. Moreover, $\tau_M(\xi)$ is multiplicative with respect to the connected sum.

Remark. If we drop the condition that the colors in the summation are odd, and sum over all (odd and even) colors, the result will be the $SU(2)$ WRT invariant $\tau_M^{SU(2)}(\xi)$. In this case, the order of ξ could also be even.

The $SO(3)$ WRT invariant extends naturally to the invariant of the pair (M, L'), where the manifold M contains a link L' inside. In this case we have to replace the surgery link L of M by $L \cup L'$ in all definitions, fix colors on L' and sum over all colorings of L only. We omit here the precise definition and refer to [2] for more details.

For example, the $SO(3)$ invariant of the lens space $L(b, 1)$, obtained by surgery along U^b, is

$$\tau_{L(b,1)}(\xi) = \frac{F_{U^b}(\xi)}{F_{U^{\mathrm{sn}(b)}}(\xi)}, \tag{4}$$

were $\mathrm{sn}(b)$ is the sign of the integer b.

Suppose M is a rational homology 3–sphere. Then there is a unique decomposition $H_1(M, \mathbb{Z}) = \bigoplus_i \mathbb{Z}/b_i\mathbb{Z}$, where each b_i is a prime power. We renormalize the $SO(3)$ WRT invariant as follows:

$$\tau'_M(\xi) = \frac{\tau_M(\xi)}{\prod_i \tau_{L(b_i,1)}(\xi)}\ , \tag{5}$$

where $L(b, a)$ denotes the (b, a) lens space. Note that $\tau_{L(b,1)}(\xi)$ is always nonzero.

Let us focus on the special case when the linking matrix of L is diagonal. Assume each $L_{ii} = b_i$ is a power of a prime or 1, up to sign. Then $H_1(M, \mathbb{Z}) = \oplus_{i=1}^m \mathbb{Z}/|b_i|$, and

$$\sigma_+ = \mathrm{card}\,\{i \mid b_i > 0\}, \quad \sigma_- = \mathrm{card}\,\{i \mid b_i < 0\}.$$

Thus from the definitions (3), (4) and (5) we have

$$\tau'_M(\xi) = \left(\prod_{i=1}^m \tau'_{L(b_i,1)}(\xi)\right) \frac{F_L(\xi)}{\prod_{i=1}^m F_{U^{b_i}}(\xi)}\ , \tag{6}$$

with

$$\tau'_{L(b_i,1)}(\xi) = \frac{\tau_{L(b_i,1)}(\xi)}{\tau_{L(|b_i|,1)}(\xi)}\ .$$

The collection of $SO(3)$ WRT invariants is difficult to study, since their definition heavily depends on the order of ξ. The following cyclotomic expansion will play an important role for the unification of the WRT invariants.

1.6. **Habiro's cyclotomic expansion of the colored Jones polynomial.** For non–negative integers n, k we define

$$A(n, k) := \frac{\prod_{i=0}^k (q^n + q^{-n} - q^i - q^{-i})}{(1-q)\,(q^{k+1}; q)_{k+1}}.$$

For $\mathbf{k} = (k_1, \ldots, k_m)$ let

$$A(\mathbf{n}, \mathbf{k}) := \prod_{j=1}^m A(n_j, k_j).$$

Note that $A(\mathbf{n}, \mathbf{k}) = 0$ if $k_j \geq n_j$ for some index j. Also $A(n, 0) = q^{-1} J_U(n)^2$.

The colored Jones polynomial $J_L(\mathbf{n})$ can be repackaged into the invariant $C_L(\mathbf{k})$ as stated in the following theorem.

Theorem 2. [Habiro] *Suppose L is a link in S^3 having zero linking matrix. Then there are invariants*

$$C_L(\mathbf{k}) \in \frac{(q^{k+1};q)_{k+1}}{(1-q)}\,\mathbb{Z}[q^{\pm 1}], \quad \textit{where } k = \max\{k_1,\dots,k_m\} \tag{7}$$

such that for every $\mathbf{n} = (n_1,\dots,n_m)$

$$J_L(\mathbf{n}) \prod_{i=1}^{m} [n_i] = \sum_{0\le k_i \le n_i-1} C_L(\mathbf{k})\, A(\mathbf{n},\mathbf{k}). \tag{8}$$

Note that the existence of $C_L(\mathbf{k})$ as rational functions in q satisfying (8) is easy to establish. They correspond to the Jones polynomial colored by different elements of the Grothendieck ring of $U_q(\mathfrak{sl}_2)$, i.e. by linear combinations of representations. The difficulty here is to show the integrality of (7).

Since $A(\mathbf{n},\mathbf{k}) = 0$ unless $\mathbf{k} < \mathbf{n}$, in the sum on the right hand side of (8) one can assume that $\mathbf{k}$ runs over the set of all m–tuples $\mathbf{k}$ with non–negative integer components. We will use this fact later.

2. Main Results

Let us state our main results announced in the introduction.

For any positive integer b, we define the cyclotomic completion ring $\mathcal{R}_b$ to be

$$\mathcal{R}_b := \varprojlim_k \frac{\mathbb{Z}[1/b][q]}{((q;q^2)_k)}, \qquad \text{where} \quad (q;q^2)_k = (1-q)(1-q^3)\dots(1-q^{2k-1}). \tag{9}$$

For any $f(q) \in \mathcal{R}_b$ and a root of unity ξ of *odd* order, the evaluation $\mathrm{ev}_\xi(f(q)) := f(\xi)$ is well–defined. Similarly, we put

$$\mathcal{S}_b := \varprojlim_k \frac{\mathbb{Z}[1/b][q]}{((q;q)_k)}\ .$$

Here the evaluation at any root of unity is well–defined. For odd b, there is a natural embedding $\mathcal{S}_b \hookrightarrow \mathcal{R}_b$.

Let us denote by $\mathcal{M}_b$ the set of rational homology 3–spheres such that $|H_1(M,\mathbb{Z})|$ divides b^n for some n. Our main result is the following.

Theorem 3. [Beliakova–Bühler–Le] *Suppose the components of a framed oriented link $L \subset M$ have odd colors, and $M \in \mathcal{M}_b$. Then there exists an invariant $I_{M,L} \in \mathcal{R}_b$, such that for any root of unity ξ of odd order*

$$\mathrm{ev}_\xi(I_{M,L}) = \tau'_{M,L}(\xi)\,.$$

In addition, if b is odd, then $I_{M,L} \in \mathcal{S}_b$.

If $b = 1$ and L is the empty link, I_M coincides with Habiro's unified invariant J_M and $\mathcal{S}_1 = \widehat{\mathbb{Z}[q]}$.

One may ask what is the evaluation of I_M at an even root of unity in the case when b is odd. In her PhD thesis [5], Bühler shows that it coincides with $\tau'^{SU(2)}_M(\xi)$. Hence, for odd b, I_M dominates both $SO(3)$ and $SU(2)$ WRT invariants. An analogous result for b even is work in progress.

Compared to Habiro's case, the proof of Theorem 3 uses the following new techniques: 1) the Laplace transform method; 2) the difficult number theoretical identity of Andrews generalizing those of Roger–Ramanujan; 3) the Frobenius type isomorphism providing the existence of the b–th root of q in $\mathcal{R}_b$. In addition, we had to generalize the deep integrality result of Habiro (Theorem 2), to a union of an algebraically split link with any odd colored one.

The rings $\mathcal{R}_b$ and $\mathcal{S}_b$ have properties similar to those of the Habiro ring. An element $f(q) \in \mathcal{R}_b$ is totally determined by the values at many infinite sets of roots of unity (see Section 3), one special case is the following.

Proposition 4. [Beliakova–Bühler–Le] *Let p be an odd prime not dividing b and T the set of all integers of the form $p^k b'$ with $k \in \mathbb{N}$ and b' any odd divisor of b^n for some n. Any element $f(q) \in \mathcal{R}_b$, and hence also $\{\tau_M(\xi)\}$, is totally determined by the values at roots of unity with orders in T.*

The general properties of the ring $\mathcal{R}_b$ allow to introduce generalized Ohtsuki series as the Taylor expansions of I_M at roots of unity. In addition, we show that these Taylor expansions satisfy congruence relations similar to the original definition of the Ohtsuki series (see Section 3.5).

2.1. **Strategy of the proof.** Let us outline the proof of Theorem 3 and state the main technical results that will be explained later.

We restrict to the case $L = \emptyset$ for simplicity. We would like to define $I_M \in \mathcal{R}_b$, such that

$$\tau'_M(\xi) \;=\; \mathrm{ev}_\xi\,(I_M) \tag{10}$$

for any root of unity ξ of odd order. This unified invariant is multiplicative with respect to the connected sum.

The following observation is important. By Proposition 4, there is *at most one* element $f(q) \in \mathcal{R}_b$ such that for every root ξ of odd order one has

$$\tau'_M(\xi) = \mathrm{ev}_\xi\,(f(q))\,.$$

That is, if we can find such an element, it is unique, and we put $I_M := f(q)$.

2.2. **Laplace transform.** The following is the main technical result of [2]. A proof will be explained in the next Section.

Theorem 5. [Beliakova–Bühler–Le] *Suppose $b = \pm 1$ or $b = \pm p^l$ where p is a prime and l is positive. For any non–negative integer k, there exists an element $Q_{b,k} \in \mathcal{R}_b$ such that*

for every root ξ of odd order r one has

$$\frac{\sum_n^\xi q^{b\frac{n^2-1}{4}} A(n,k)}{F_{U^b}(\xi)} = \mathrm{ev}_\xi(Q_{b,k}).$$

In addition, if b is odd, $Q_{b,k} \in \mathcal{S}_b$.

2.3. **Definition of the unified invariant: diagonal case.** Suppose that the linking number between any two components of L is 0, and the framing on components of L are $b_i = \pm p_i^{k_i}$ for $i = 1, \dots, m$, where each p_i is prime or 1. Let us denote the link L with all framings switched to zero by L_0.

Using (8), taking into account the framings b_i's, we have

$$J_L(\mathbf{n}) \prod_{i=1}^m [n_i] = \sum_{\mathbf{k} \geq 0} C_{L_0}(\mathbf{k}) \prod_{i=1}^m q^{b_i \frac{n_i^2-1}{4}} A(n_i, k_i).$$

By the definition of F_L, we have

$$F_L(\xi) = \sum_{\mathbf{k} \geq 0} \mathrm{ev}_\xi(C_{L_0}(\mathbf{k})) \prod_{i=1}^m \sum_{n_i}^\xi q^{b_i \frac{n_i^2-1}{4}} A(n_i, k_i).$$

From (6) and Theorem 5, we get

$$\tau'_M(\xi) = \mathrm{ev}_\xi \left\{ \prod_{i=1}^m I_{L(b_i,1)} \sum_{\mathbf{k}} C_{L_0}(\mathbf{k}) \prod_{i=1}^m Q_{b_i,k_i} \right\},$$

where the existence of the unified invariant of the lens space $I_{L(b_i,1)} \in \mathcal{R}_b$, with $\mathrm{ev}_\xi(I_{L(b_i,1)}) = \tau'_{L(b_i,1)}(\xi)$ can be shown by a direct computation (we refer to [2] for more details).

Thus if we define

$$I_M := \prod_{i=1}^m I_{L(b_i,1)} \sum_{\mathbf{k}} C_{L_0}(\mathbf{k}) \prod_{i=1}^m Q_{b_i,k_i},$$

then (10) is satisfied. By Theorem 2, $C_{L_0}(\mathbf{k})$ is divisible by $(q^{k+1};q)_{k+1}/(1-q)$, which is divisible by $(q;q)_k$, where $k = \max k_i$. It follows that $I_M \in \mathcal{R}_b$. In addition, if b is odd, then $I_M \in \mathcal{S}_b$.

2.4. **Diagonalization using lens spaces.** The general case reduces to the diagonal case by the well–known trick of diagonalization using lens spaces. We say that M is *diagonal* if it can be obtained from S^3 by surgery along a framed link L with diagonal linking matrix, where the diagonal entries are of the form $\pm p^k$ with $p = 0, 1$ or a prime. The following lemma was proved in [12, Proposition 3.2 (a)].

Lemma 6. *For every rational homology sphere M, there are lens spaces $L(b_i, a_i)$ such that the connected sum of M and these lens spaces is diagonal. Moreover, each b_i is a prime power divisor of $|H_1(M,\mathbb{Z})|$.*

To define the unified invariant for a general rational homology sphere M, one first adds to M lens spaces to get a diagonal M', for which the unified invariant $I_{M'}$ had been defined in Subsection 2.3. Then I_M is the quotient of $I_{M'}$ by the unified invariants of the lens spaces. But unlike the simpler case of [12], the unified invariant of lens spaces are *not*

invertible in general. To overcome this difficulty we insert knots in lens spaces and split the unified invariant into different components. This is also the reason why we need to generalize Habiro's integrality result to algebraically split links together with odd colored components.

3. Cyclotomic completions of polynomial rings

Since unified invariants belong to cyclotomic completions of polynomial rings, we outline their construction and the main properties. For simplicity, only the case b is a power of a prime is considered, the general case is treated in [2].

3.1. **On cyclotomic polynomials.** Recall that $e_n := \exp(2\pi I/n)$ and denote by $\Phi_n(q)$ the cyclotomic polynomial

$$\Phi_n(q) = \prod_{\substack{(j,n)=1\\0<j<n}} (q - e_n^j).$$

For example, $\Phi_1(q) = q - 1$ and $\Phi_2(q) = q + 1$. The degree of $\Phi_n(q) \in \mathbb{Z}[q]$ is given by the Euler function $\varphi(n)$. Suppose p is a prime and n an integer. Then (see e.g. [18])

$$\Phi_n(q^p) = \begin{cases} \Phi_{np}(q) & \text{if } p \mid n \\ \Phi_{np}(q)\Phi_n(q) & \text{if } p \nmid n. \end{cases} \tag{11}$$

It follows that $\Phi_n(q^p)$ is always divisible by $\Phi_{np}(q)$.

The ideal of $\mathbb{Z}[q]$ generated by $\Phi_n(q)$ and $\Phi_m(q)$ is well–known, see e.g. [12, Lemma 5.4]:

Lemma 7.

(a) *If $\frac{m}{n} \neq p^e$ for any prime p and any integer $e \neq 0$, then $(\Phi_n) + (\Phi_m) = (1)$ in $\mathbb{Z}[q]$.*
(b) *If $\frac{m}{n} = p^e$ for a prime p and some integer $e \neq 0$, then $(\Phi_n) + (\Phi_m) = (p)$ in $\mathbb{Z}[q]$.*

3.2. **Habiro's results.** Let us summarize some of Habiro's results on cyclotomic completions of polynomial rings [8]. Let R be a commutative integral domain of characteristic zero and $R[q]$ the polynomial ring over R. For any $S \subset \mathbb{N}$, Habiro defined the S–cyclotomic completion ring $R[q]^S$ as follows:

$$R[q]^S := \varprojlim_{f(q)\in\Phi_S^*} \frac{R[q]}{(f(q))} \tag{12}$$

where Φ_S^* denotes the multiplicative set in $\mathbb{Z}[q]$ generated by $\Phi_S = \{\Phi_n(q) \mid n \in S\}$ and directed with respect to the divisibility relation.

For example, since the sequence $(q;q)_n$, $n \in \mathbb{N}$, is cofinal to $\Phi_{\mathbb{N}}^*$, we have

$$\widehat{\mathbb{Z}[q]} \simeq \mathbb{Z}[q]^{\mathbb{N}}. \tag{13}$$

Note that if S is finite, then $R[q]^S$ is identified with the $(\prod \Phi_S)$–adic completion of $R[q]$. In particular,

$$R[q]^{\{1\}} \simeq R[[q-1]], \quad R[q]^{\{2\}} \simeq R[[q+1]].$$

Suppose $S' \subset S$, then $\Phi^*_{S'} \subset \Phi^*_S$, hence there is a natural map

$$\rho^R_{S,S'} : R[q]^S \to R[q]^{S'}.$$

Recall important results concerning $R[q]^S$ from [8]. Two positive integers n, n' are called *adjacent* if $n'/n = p^e$ with a nonzero $e \in \mathbb{Z}$ and a prime p, such that the ring R is p–adically separated, i.e. $\bigcap_{n=1}^\infty (p^n) = 0$ in R. A set of positive integers is R*–connected* if for any two distinct elements n, n' there is a sequence $n = n_1, n_2, \dots, n_{k-1}, n_k = n'$ in the set, such that any two consecutive numbers of this sequence are adjacent. Theorem 4.1 of [8] says that if S is R–connected, then for any subset $S' \subset S$ the natural map $\rho^R_{S,S'} : R[q]^S \hookrightarrow R[q]^{S'}$ is an embedding.

Example: Assume $R = \mathbb{Z}$, $S = \mathbb{N}$ and $S' = \{1\}$, then we have that the map $\widehat{\mathbb{Z}[q]} \to \mathbb{Z}[[q-1]]$ is an embedding. This implies that, for any integral homology 3–sphere M, J_M is determined by the Ohtsuki series and this series has integral coefficients.

If ζ is a root of unity of order in S, then for every $f(q) \in R[q]^S$ the evaluation $\mathrm{ev}_\zeta(f(q)) \in R[\zeta]$ can be defined by sending $q \to \zeta$. For a set Ξ of roots of unity whose orders form a subset $\mathcal{T} \subset S$, one defines the evaluation

$$\mathrm{ev}_\Xi : R[q]^S \to \prod_{\zeta \in \Xi} R[\zeta].$$

Theorem 6.1 of [8] shows that if $R \subset \mathbb{Q}$, S is R–connected, and there exists $n \in S$ that is adjacent to infinitely many elements in $\mathcal{T}$, then ev_Ξ is injective.

Example: Consider again the case when $R = \mathbb{Z}$, $S = \mathbb{N}$ and put $\mathcal{T} = \{3^n | n \in \mathbb{N}\}$, then $3 \in S$ is adjacent to infinitely many elements of $\mathcal{T}$ and hence, for any integral homology 3–sphere M, the whole set of its WRT invariants is determined by the evaluations of J_M at roots of unity of order in $\mathcal{T}$.

3.3. **Taylor expansion.** Fix a natural number n, then we have

$$R[q]^{\{n\}} = \varprojlim_k \frac{R[q]}{(\Phi^k_n(q))}.$$

Suppose $\mathbb{Z} \subset R \subset \mathbb{Q}$, then the natural algebra homomorphism

$$h : \frac{R[q]}{(\Phi^k_n(q))} \to \frac{R[e_n][q]}{((q-e_n)^k)}$$

can be proved to be injective. Taking the inverse limit, we see that there is a natural injective algebra homomorphism

$$h : R[q]^{\{n\}} \to R[e_n][[q-e_n]].$$

Suppose $n \in S$. Combining h and $\rho_{S,\{n\}} : R[q]^S \to R[q]^{\{n\}}$, we get an algebra map

$$\mathfrak{t}_n : R[q]^S \to R[e_n][[q-e_n]].$$

If $f \in R[q]^S$, then $\mathfrak{t}_n(f)$ is called the Taylor expansion of f at e_n.

3.4. **Splitting of $\mathcal{S}_p$ and evaluation.** For every integer a, we put $\mathbb{N}_a := \{n \in \mathbb{N} \mid (a,n) = 1\}$.

Suppose p is a prime. Analogously to (13), we have

$$\mathcal{S}_p \simeq \mathbb{Z}[1/p][q]^{\mathbb{N}}.$$

Observe that $\mathbb{N}$ is not $\mathbb{Z}[1/p]$–connected. In fact one has $\mathbb{N} = \amalg_{j=0}^{\infty} p^j \mathbb{N}_p$, where each $p^j\mathbb{N}_p$ is $\mathbb{Z}[1/p]$–connected. Let us define

$$\mathcal{S}_{p,j} := \mathbb{Z}[1/p][q]^{p^j\mathbb{N}_p}.$$

Note that for every $f \in \mathcal{S}_p$, the evaluation $\mathrm{ev}_\xi(f)$ can be defined for every root ξ of unity. For $f \in \mathcal{S}_{p,j}$, the evaluation $\mathrm{ev}_\xi(f)$ can be defined when ξ is a root of unity of order in $p^j\mathbb{N}_p$. In [2] we proved the following.

Proposition 8. *For every prime p one has*

$$\mathcal{S}_p \simeq \prod_{j=0}^{\infty} \mathcal{S}_{p,j}. \tag{14}$$

Let $\pi_j : \mathcal{S}_p \to \mathcal{S}_{p,j}$ denote the projection onto the jth component in the above decomposition. Suppose ξ is a root of unity of order $r = p^j r'$, with $(r', p) = 1$. Then for any $x \in \mathcal{S}_p$, one has

$$\mathrm{ev}_\xi(x) = \mathrm{ev}_\xi(\pi_j(x)).$$

If $i \neq j$ then $\mathrm{ev}_\xi(\pi_i(x)) = 0$.

3.5. **On the Ohtsuki series at roots of unity.** Suppose M is a rational homology 3–sphere with $|H_1(M,\mathbb{Z})| = b$. By Theorem 3, for any root of unity ξ of order pr

$$\tau'_M(\xi) \in \mathbb{Z}[1/b][e_{pr}] \simeq \frac{\mathbb{Z}[1/b, e_r][x]}{(f_p(x+e_r))}.$$

where

$$f_p(t) := \frac{t^p - e_r^p}{t - e_r}.$$

Hence we can write

$$\tau'_M(e_r e_p) = \sum_{n=0}^{p-2} a_{p,n} x^n \tag{15}$$

where $a_{p,n} \in \mathbb{Z}[1/b, e_r]$. The following proposition proven in [2] shows that the coefficients $a_{p,n}$ stabilize as $p \to \infty$.

Proposition 9. [Beliakova–Bühler–Le] *Suppose M is a rational homology 3–sphere with $|H_1(M,\mathbb{Z})| = b$, and r is an odd positive integer. For every non–negative integer n, there exists a unique invariant $a_n = a_n(M) \in \mathbb{Z}[1/b, e_r]$ such that for every prime $p > \max(b,r)$, we have*

$$a_n \equiv a_{p,n} \pmod p \quad \textit{in } \mathbb{Z}[1/b, e_r] \textit{ for } \; 0 \le n \le p-2. \tag{16}$$

Moreover, the formal series $\sum_n a_n (q-e_r)^n$ is equal to the Taylor expansion of the unified invariant I_M at e_r.

4. Laplace transform, Andrews identity and Frobenius maps

The aim of this section is to define the Laplace transform and to study its image.

4.1. **Laplace transform.** To define the unified invariant we have to compute

$$\sum_n^{\xi} q^{b\frac{n^2-1}{4}} A(n,k)$$

where terms depending on n in $A(n,k)$ look as follows

$$\prod_{i=0}^{k}(q^n+q^{-n}-q^i-q^{-i}) = (-1)^{k+1}\,(q^n;q)_{k+1}\,(q^{-n};q)_{k+1}\,.$$

Formally, the last expression can be considered as a Laurent polynomial in q and q^n. Hence we only have to compute

$$\sum_n^{\xi} q^{b\frac{n^2-1}{4}} q^{na}\,,$$

which can be easily done by the square completing argument. Let us state the result.

Suppose r is an odd number, b is positive integer and

$$c := (r,b), \quad b_1 := b/c, \quad r_1 := r/c.$$

Lemma 10. [Beliakova–Le] *One has*

$$\gamma_b(\xi) = c\,\gamma_{b_1}(\xi^c). \tag{17}$$

$$\sum_n^{\xi} q^{b\frac{n^2-1}{4}}\, q^{an} = \begin{cases} 0 & \text{if } c \nmid a; \\ (\xi^c)^{-a_1^2 b_1^*}\,\gamma_b(\xi) & \text{if } a = ca_1, \end{cases} \tag{18}$$

where b_1^* is an integer satisfying $b_1 b_1^* \equiv 1 \pmod{r_1}$.

This computation inspired us to introduce the following operator, called the Laplace transform. Remember that $\int e^{ax^k} f(x)\,dx$ is called the Laplace transform of f of order k.

Let $\mathcal{L}_{b,c;n} : \mathbb{Z}[q^{\pm n}, q^{\pm 1}] \to \mathbb{Z}[q^{\pm c/b}]$ be the $\mathbb{Z}[q^{\pm 1}]$–linear operator, called the (discrete) Laplace transform (of the second order), defined by

$$\mathcal{L}_{b,c;n}(q^{na}) := \begin{cases} 0 & \text{if } c \nmid a; \\ q^{-a^2/b} & \text{if } a = ca_1, \end{cases} \tag{19}$$

Lemma 11. [Beliakova–Le] *Suppose $f \in \mathbb{Z}[q^{\pm n}, q^{\pm 1}]$. Then*

$$\sum_n^{\xi} q^{b\frac{n^2-1}{4}} f = \gamma_b(\xi)\,\mathrm{ev}_\xi(\mathcal{L}_{b,c;n}(f)).$$

The point is that $\mathcal{L}_{b,c;n}(f)$, unlike the left hand side $\sum_n^{\xi} q^{b\frac{n^2-1}{4}} f$, does not depend on ξ. To prove Theorem 5 we need to show that $\mathcal{L}_{b,c;n}((q^n;q)_{k+1}\,(q^{-n};q)_{k+1})$ is divisible by $(q^{k+1};q)_{k+1}$. For this we use the remarkable identity discovered by Andrews.

4.2. **Andrews identity.** To warm up we start with the identity from the Ramanujan "Lost" Notebook:

$$\prod_{k\geq 1}\frac{1}{(1-q^{5k-4})(1-q^{5k-1})}=\sum_{n\geq 0}\frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)}$$

Only in the 60s, MacMahon gave a combinatorial interpretation of this identity as follows. Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_t)$ with $\sum_i\lambda_i=n$ be a partition of n into non–increasing integers. Then the identity can derived from the fact that the number of partitions of n with all λ_i of the form $5k+1$ or $5k+4$ is equal to the number of partitions where $\lambda_i-\lambda_{i+1}\geq 2$ for all i.

This identity is a very special case of the Andrews identity we used in [2]: For any numbers $b_i,c_i,i=1,\dots,k$ and positive integer N we have

$$1+\sum_{n=1}^{N}q^{kn+Nn}(1+q^n)\frac{(q^{-N})_n}{(q^{N+1})_n}\prod_{i=1}^{k}\frac{(b_i)_n}{b_i^n}\frac{(c_i)_n}{c_i^n}\frac{1}{(\frac{q}{b_i})_n(\frac{q}{c_i})_n}=$$

$$\frac{(q)_N\,(\frac{q}{b_kc_k})_N}{(\frac{q}{b_k})_N\,(\frac{q}{c_k})_N}\sum_{n_k\geq n_{k-1}\geq\cdots\geq n_2\geq n_1=0}\frac{q^{n_k}(q^{-N})_{n_k}(b_k)_{n_k}(c_k)_{n_k}}{(q^{-N}b_kc_k)_{n_k}}\prod_{i=1}^{k-1}\frac{q^{n_i}\frac{(b_i)_{n_i}}{b_i^{n_i}}\frac{(c_i)_{n_i}}{c_i^{n_i}}(\frac{q}{b_ic_i})_{n_{i+1}-n_i}}{(q)_{n_{i+1}-n_i}(\frac{q}{b_i})_{n_{i+1}}(\frac{q}{c_i})_{n_{i+1}}}$$

where $(a)_n=(a;q)_n$.

The point is that (for a special choice of parameters) the left hand side of the identity can be identified with $\mathcal{L}_{b,c;n}\left((q^n;q)_{k+1}\,(q^{-n};q)_{k+1}\right)$, where the right hand side is a sum with all summands divisible by $(q^{k+1};q)_{k+1}$.

In the case $b=\pm 1$, the computations are especially simple and we get that

$$\mathcal{L}_{-1,1;n}((q^n;q)_{k+1}\,(q^{-n};q)_{k+1})=2(q^{k+1};q)_{k+1}\,.$$

The same holds also for $\mathcal{L}_{1,1;n}$ up to units. This allows to write the explicit formula for I_M given in the introduction.

4.3. **Frobenius isomorphism.** It remains to show that the image of the Laplace transform belongs to $\mathcal{R}_b$, i.e. that certain roots of q exist in $\mathcal{R}_b$.

In [2] we proved the following.

Theorem 12. [Beliakova–Bühler–Le] *The Frobenius endomorphism $F_b:\mathbb{Z}[1/b][q]^{\mathbb{N}_b}\to\mathbb{Z}[1/b][q]^{\mathbb{N}_b}$, sending q to q^b, is an isomorphism.*

This implies the existence of the bth root of q in $\mathcal{S}_{b,0}$ defined by

$$q^{1/b}:=F_b^{-1}(q)\in\mathcal{S}_{b,0}\,.$$

Let us mention that this result does not hold over $\mathbb{Q}$, i.e. for $y\in\mathbb{Q}^{\mathbb{N}_b}$ with $y^b=1$ we have $y=\pm 1$.

Further, we introduce another Frobenius homomorphism

$$G_m:R[q]^{\mathbb{N}_b}\to R[q]^{m\mathbb{N}_b}\qquad\text{by}\qquad G_m(q)=q^m.$$

Since $\Phi_{mr}(q)$ always divides $\Phi_r(q^m)$, G_m is well–defined.

This map allows us to transfer $q^{1/b}$ from $\mathcal{S}_{p,0}$ to $\mathcal{S}_{p,i}$ with $i>0$, and hence to define a realization of $q^{a^2/b}$ in $\mathcal{S}_p$ with the correct evaluation.

4.4. **Integrality.** The proof of integrality of the SO(3) WRT invariants for all roots of unity and for all 3–manifolds is given in [4]. Note that even restricted to rational homology 3–spheres, this fact requires a separate proof, since the existence of $I_M \in \mathcal{R}_b$ does not imply integrality of the WRT invariants, unless $b = 1$. In this subsection we define a subring $\Gamma_b \subset \mathcal{R}_b$, such that for any $f \in \Gamma_b$, $\mathrm{ev}_\xi(f) \in \mathbb{Z}[\xi]$. The fact that $I_{M,1}$ belongs to this subring was proved in [4]. We conjecture that this holds in general.

For any divisor c of b, let us decompose $\mathbb{N} = \cup_{c|b}\, c\, \mathbb{N}_{b/c}$. Then

$$\mathbb{Z}[1/b][q]^{\mathbb{N}} = \prod_{c|b} \mathbb{Z}[1/b][q]^{c\,\mathbb{N}_{b/c}}\,.$$

Analogously, we have $\Gamma_b = \prod_{c|b} \Gamma_{b,c}$, where $\Gamma_{b,c} \subset \mathbb{Z}[1/b][q]^{c\,\mathbb{N}_{b/c}}$ is defined as follows.

Let $A_{b,c} = \mathbb{Z}[e_c][q^{\pm 1}, q^{\pm c/b}]$. Put $t = q^{c/b}$ and let $A^{(m)}_{b,c}$ be the algebra generated over $A_{b,c}$ by

$$\frac{(t;t)_m}{(q^c;q^c)_m}\;\frac{(e_c;e_c)_{(c-1)/2}}{\widetilde{(q;q)_m}}$$

where

$$\widetilde{(q;q)_m} = \prod_{i=1, c\nmid i}^{m} (1-q^i)\,.$$

Then every element $f \in \Gamma_{b,c}$ has a presentation

$$f = \sum_{m=0}^{\infty} f_m \,\frac{(q^{m+1};q)_{m+1}}{1-q}$$

with $f_m \in A^{(m)}_{b,c}$. For any root of unity ξ of odd order r with $(r,b) = c$ and $f \in \Gamma_{b,c}$, we have

$$\mathrm{ev}_\xi(f) = \mathrm{ev}_\xi\left(\sum_{m=0}^{(r-3)/2} f_m \frac{(q^{m+1};q)_{m+1}}{1-q}\right)\,.$$

Observe that for $m < (r-1)/2$,

$$\mathrm{ev}_\xi(\widetilde{(q;q)_m}) \;\mid\; (e_c;e_c)_{(c-1)/2} \quad\text{and}\quad \mathrm{ev}_\xi\left(\frac{(t;t)_m}{(q^c;q^c)_m}\right) \in \mathbb{Z}[\xi]\,.$$

Conjecture. *For any $M \in \mathcal{M}_b$, there exists an invariant $I'_M \in \Gamma_b$, such that for any root of unity ξ of odd order* $\mathrm{ev}_\xi(I'_M) = \tau_M(\xi)$.

We also expect that $\Gamma_{b,b}$ is determined by its Taylor expansion at e_b.

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